MATH3090 Final Project - Black-Scholes-Merton Formula European Call Option Pricing

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A *financial derivative* is a financial product whose value is derived from the value/price of another underlying entity or variable - the so called underlying. Financial derivatives have existed since the earliest days modern finance, dating to the late 17th century in a recognisable form similar to today, and has since grown to a more than $500 trillion market. Derivatives can be written on almost any asset you can think of, from stocks to even the weather. In this project, you will have the opportunity to study *otpions* and learn how they are priced using the Black-Scholes-Merton formula.

An *option* is a financial derivative that grants the holder the right, *but not the obligation*, to buy or sell (*exercise*) the underlying asset at a future date (the *maturity*) for an agreed upon price (the *strike*). A *call* option allows the holder to buy the underlying at a future data, while a *put* option gives the holder the right to sell the underlying. The most famous types of options are *European* and *American* stock options. European options only allow the holder to buy/sell the underlying at the agreed upon price at the date in the future - the *maturity*. American options, on the other hand, allow the holder to buy/sell the underlying for the agreed upon price at any date up to and including the maturity.

As options give you the ability to choose (the *option* to choose) whether or not to sell for the agreed upon price, you should have to pay a price for this ability. But what is a fair price to pay? The answer is given by the Black-Scholes-Merton formula [1, 2, 3], a hallmark of financial economics, and perhaps the most commonly computed formula on a daily basis around the world.

We will focus exclusively on computing the price of European options, specifically computing solving the Black-Scholes-Merton equation to compute the price of European call options. Due to the added complexity of deciding when to exercise American options, the pricing of American options remains an active area of research. Formally deriving the Black-Scholes-Merton equation is beyond the scope of this course and requires knowledge of stochastic calculus.

We consider the following factors:

1. Current stock price, S_0 or S_t - a function of time
2. The strike price, K - what we will buy/sell for at maturity
3. Time to expiration/maturity, T
4. Volatility of the stock price, σ - how much the stock price varies
5. Risk-free interest rate, r - what your money will earn in the bank

To derive the Black-Scholes-Merton equation, Black, Scholes, and Merton made the following assumptions on the market and economy

1. Stock returns are log-normally distributed (this is just a modelling assumption)
   - The amount made or lost on stock price gains or losses follows a log-normal distribution, that is, the logarithm of stock returns is normally distributed.
2. Short selling (selling stocks you don't currently have) is permitted - proceeds may be used immediately
   - Short selling involves selling stocks you don't own, but rather borrow, to make a gain. The owner of the borrowed shares will demand the shares back in the future, upon which the short seller must return them (through purchase of the shares and returning to the owner)
3. No transaction costs, all securities are divisible (e.g. buying half a stock)
   - Transaction costs refer to a fee that must be paid to buy or sell a share. Divisible securities allow one to buy or sell fractions of a share.
4. No dividends (stocks don't pay for gains)

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- A dividend is a payment made by the firm who has issued the shares for performance from profit. The amount paid by the divided is reduced off the share price.

5. No arbitrage (no risk-free profit can be made)

   - Some times called *no free lunch* - the absence of arbitrage means that there is no opportunity to make a profit without being exposed to some downside risk

6. Continuous trading (you may buy and sell at any time)

   - Shares may be purchased, sold, or shorted, at any time up to expiration

7. Risk-free interest rate, r, constant and same for all maturities.

   - The risk-free interest rate is the percentage return on money that is kept in the bank. This is the return that is made without taking on any risk.

Let f be the price of the option. f is then the solution to the Black-Scholes-Merton formula,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{1}$$

The boundary conditions of the equation are involved in determining what sort of option price is computed as the solution to the Black-Scholes-Merton equation. Since we will be pricing European call options, the boundary conditions are,

$$f(T,S) = \max\{S_T - K, 0\} = (S_T - K)^+ \tag{2}$$
$$f(t,0) = 0 \tag{3}$$
$$f(t,S) \to S - Ke^{-r(T-t)} \text{ as } S \to \infty. \tag{4}$$

To solve this equation numerically, we will use an explicit numerical scheme. Note that an explicit scheme requires you to be careful in choosing your discretization, due to the stability condition given by the Courant-Friedrichs-Lewy (CFL) condition.

We form the computational discrete by setting the option maturity at T, and S_{max} a sufficiently large asset price that cannot be reached within the time horizon we consider. The consideration of S_{max} is a subtle numerical point, as the domain of the PDE is unbounded with respect to S, but we must impose a bound for computation. Set the grid as,

$$t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t = T \tag{5}$$
$$S = 0, \Delta S, 2\Delta S, \ldots, M\Delta S = S_{max} \tag{6}$$

with grid notation $f_{i,j} = f(i\Delta t, j\Delta S)$.

We have to appropriately adapt the boundary conditions for computation. When the asset price is $S(t) = 0$, for any t, the option is worthless,

$$f(t,0) = 0. \tag{7}$$

For large asset prices, $S(t)$, the option is definitely in-the-money (you make money by exercising it) at expiration, getting payoff $S(T) - K$. The value at time t requires discounting back the strike K at the risk-free interest rate, and using the arbitrage free price of the underlying is $S(t)$. A suitable boundary condition is then,

$$f(t, S_{\max}) = S_{max} - Ke^{-r(T-t)}. \tag{8}$$

Discretized, these take the form

$$f_{N,j} = (j\Delta S - K)^+, \; j = 0, 1, \ldots, M \tag{9}$$
$$f_{i,0} = 0, \; i = 0, 1, \ldots N \tag{10}$$
$$f_{i,M} = M\Delta S - Ke^{-r(N-i)\Delta t}, \; i = 0, 1, \ldots, N. \tag{11}$$

The discretized PDE is

$$\frac{f_{i,j} - f_{i-1,j}}{\Delta t} + rj\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} = rf_{i,j}. \tag{12}$$

The unknown in this equation is $f_{i-1,j}$. Due to the form of the equation, you will solve this equation backward in time, which is specified by using a backward difference method in time. This uses the solutions from forward in time, and works backwrds. Rearranging we have,

$$f_{i-1,j} = a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} \tag{13}$$

$$a_j = \frac{1}{2}\Delta t(\sigma^2 j^2 - rj) \tag{14}$$

$$b_j = 1 - \Delta t(\sigma^2 j^2 + r) \tag{15}$$

$$c_j = \frac{1}{2}\Delta t(\sigma^2 j^2 + rj). \tag{16}$$

Specify a collection of initial prices S_0, risk-free interest rates, r, strike prices K, volatilities σ, and maturities, T, and compute the price of European call options. Comment on how the computed call option price changes as the conditions you supply change; for example, how does the price change as you vary the strike? If the initial underlying price, S_0, does not lie on one of the grid points, you will have to use a linear interpolation between the computed price of the option and the initial price, S_0, with the two neighbouring points to get the price.

Black and Scholes [1] were also able to work out an analytic solution to the PDE for pricing calls. The analytic solution is given by,

$$f = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{17}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{18}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \tag{19}$$

(20)

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. Compare your price computed from the numerical approximation to the price computed from the analytic solution of the Black-Scholes-Merton equation. Discuss the error between the prices.

References

[1] Black, F. and M. Scholes. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637-59.

[2] Merton, R.C. "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Sceince*, 4 (Spring 1973): 141-83.

[3] Merton, R.C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29, 2 (1974): 449-70.